DUAL SEMIGROUPS AND SECOND ORDER LINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS

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ABSTRACT

It is shown that general second order elliptic boundary value problems on bounded domains generate analytic semigroups on L_1. The proof is based on Phillips' theory of dual semigroups. Several sharp estimates for the corresponding semigroups in L_p, $1 \leqq p < \infty$, are given.

Introduction

Throughout this paper Ω denotes a bounded domain in $\mathbf{R}^n$ of class C^2. Thus $\partial\Omega$ is an $(n-1)$-dimensional C^2-manifold such that Ω lies locally on one side of $\partial\Omega$. Moreover we suppose that $\partial\Omega = \Gamma_0 \cup \Gamma_1$, where Γ_0 and Γ_1 are both open and closed in $\partial\Omega$ and $\Gamma_0 \cap \Gamma_1 = \emptyset$.

We consider *regular elliptic boundary value problems* $(\mathcal{A}, \mathcal{B})$ with real continuous coefficients on Ω, that is,

$$\mathcal{A}u := -D_j(a_{jk}D_ku) + a_jD_ju + a_0u,$$

where $a_{jk} = a_{kj} \in C^1(\bar{\Omega}, \mathbf{R})$, $a_j, a_0 \in C(\bar{\Omega}, \mathbf{R})$,

$$a_{jk}(x)\xi^j\xi^k > 0 \qquad \forall x \in \bar{\Omega}, \quad \xi = (\xi^1, \cdots, \xi^n) \in \mathbf{R}^n \setminus \{0\},$$

and

$$\mathcal{B}u := \begin{cases} u & \text{on } \Gamma_0, \\ \dfrac{\partial u}{\partial \beta} + \beta_0 u & \text{on } \Gamma_1, \end{cases}$$

where $\beta \in C^1(\Gamma_1, \mathbf{R}^n)$ is an outward pointing, nowhere tangent vector field and $\beta_0 \in C^1(\Gamma_1, \mathbf{R})$. (We use the summation convention throughout.) Thus $\mathcal{B}$ is the

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Dirichlet boundary operator on Γ_0 and the Neumann or a regular oblique derivative boundary operator on Γ_1. Of course, either Γ_0 or Γ_1 may be empty.

This boundary value problem (BVP) induces a linear operator

$$A_0 : C^2_{\mathscr{B}}(\bar{\Omega}) := \{u \in C^2(\bar{\Omega}) \mid \mathscr{B}u = 0\} \to C(\bar{\Omega})$$

by letting $A_0 u := \mathscr{A}u$. Considered as an unbounded linear operator in $L_p(\Omega)$, $1 \leqq p < \infty$, A_0 is closable and densely defined. The closure of A_0 in $L_p(\Omega)$ is denoted by A_p and said to be the *L_p-realization of the BVP* $(\mathscr{A}, \mathscr{B})$.

It is known (e.g. [11, 33]) that $-A_p$ generates an analytic semigroup in $L_p(\Omega)$ for $1 < p < \infty$. It is the main purpose of this paper to prove — under slightly stronger regularity hypotheses — that the same is true in $L_1(\Omega)$. In fact, in a natural (formal) way one can associate with the BVP $(\mathscr{A}, \mathscr{B})$ a *formally adjoint BVP* $(\mathscr{A}^\#, \mathscr{B}^\#)$ which is, up to regularity, of the same form as $(\mathscr{A}, \mathscr{B})$. Hence, we impose additional regularity restrictions on $(\mathscr{A}, \mathscr{B})$ and Γ_1 if we suppose that $(\mathscr{A}^\#, \mathscr{B}^\#)$ is a regular elliptic BVP (see Section 4 for details).

Our main result is the following:

Suppose that $(\mathscr{A}^\#, \mathscr{B}^\#)$ is a regular elliptic BVP. Then $-A_1$ generates a compact, positive, analytic semigroup on $L_1(\Omega)$.

In addition we study the question under what conditions $-A_1$ generates a contraction semigroup on $L_1(\Omega)$ and when $-A_p$ generates a contraction semigroup on $L_p(\Omega)$ for each $p \in [1, \infty)$. These results generalize earlier theorems due to Brézis and Strauss [6] (cf. also [3]).

The fundamental idea of this paper is based on a slight extension of R. S. Phillips' theory of dual semigroups [24], which is interesting for its own sake. Using this theory it is shown that we can associate with $L_1(\Omega)$ a "semigroup dual" $L_1^\bullet(\Omega)$, given by

$$C_0(\bar{\Omega}) := \{u \in C(\bar{\Omega}) \mid u \mid \Gamma_0 = 0\},$$

such that the semigroup dual $C_0^\bullet(\bar{\Omega})$ of $C_0(\bar{\Omega})$ is again $L_1(\Omega)$. Thus, $L_1^{\bullet\bullet}(\Omega) = L_1(\Omega)$, that is, $L_1(\Omega)$ is "semigroup reflexive". Moreover, with A_1 we can associate a "semigroup dual" $A_1^\bullet$ in $L_1^\bullet(\Omega)$ such that $A_1^{\bullet\bullet} = A_1$, and it can be shown that $A_1^\bullet$ is induced in a natural way by the formally adjoint BVP $(\mathscr{A}^\#, \mathscr{B}^\#)$ in $C_0(\bar{\Omega})$.

On the basis of this knowledge we study regular elliptic BVPs in $C_0(\bar{\Omega})$. Due to recent results of Stewart [30, 31] it follows that $-A_1^\bullet$ generates an analytic semigroup on $C_0(\bar{\Omega})$. By means of the general semigroup duality we obtain from this fact easily the stated result for $-A_1$. Similarly we deduce by this method the fact that $-A_1$ generates (given additional hypotheses) a contraction semigroup on $L_1(\Omega)$ from the corresponding result in $C_0(\bar{\Omega})$, where it is an almost trivial consequence of the maximum principle. Finally, the general case $1<p<\infty$ is treated by interpolation.

The functional analytic abstract framework is given in Sections 2 and 3. Our main results about semigroups generated by the BVP $(\mathscr{A}, \mathscr{B})$ are contained in Sections 10–12. An important technical lemma about an appropriate right-inverse of the boundary operator $\mathscr{B}$ is proven in Section 5. The remaining sections contain studies of the BVP $(\mathscr{A}, \mathscr{B})$ in different function spaces which are important for the proofs of our principal results.

After this paper had been completed the author became aware of a paper by Tanabe [32] where it has been stated (in much greater generality but essentially without proofs) that $-A_1$ generates an analytic semigroup in $L_1(\Omega)$. Tanabe's approach is based on estimates of the Green kernel and is completely different from ours. Also Professor Pazy informed the author that in his expanded version of [23] (which is to appear as a book in Springer-Verlag) he has a proof of the fact that regular elliptic BVPs of arbitrary even order generate analytic semigroups in $L_1(\Omega)$. His proof is closely related to ours and even simpler, since he doesn't use dual semigroups explicitly. However, since our paper contains additional precise information (cf. in particular the results of Sections 11 and 12) and since our method might also be useful in other circumstances, the publication of this paper seems to be justified.

1. Preliminaries

Let X be a Banach space over $\mathbf{K}(:=\mathbf{R}$ or $\mathbf{C})$. Then X' denotes the dual of X and $\langle .,. \rangle : X' \times X \to \mathbf{K}$ the duality pairing. If Y is a locally convex topological vector space over the same field, $\mathscr{L}(X, Y)$ is the space of all continuous linear operators from X into Y, endowed with the topology of uniform convergence on bounded subsets of X. Thus $\mathscr{L}(X, Y)$ is a Banach space with respect to the standard operator norm if Y is a Banach space, and $\mathscr{L}(X) := \mathscr{L}(X, X)$. We denote by $\mathscr{K}(X, Y)$ the closed linear subspace of $\mathscr{L}(X, Y)$ consisting of all compact linear operators, and $\mathscr{K}(X) := \mathscr{K}(X, X)$. We write $X \hookrightarrow Y$ or $X \subset\hookrightarrow Y$, respectively, if X is a linear subspace of Y and the natural injection is continuous or compact, respectively.

Let $X \hookrightarrow Y$ and let $A : \mathrm{dom}(A) \subset Y \to Y$ be linear. Then we define the *X-realization* A_X of A,

$$A_X : \mathrm{dom}(A_X) \subset X \to X,$$

by $\mathrm{dom}(A_X) := \{x \in X \cap \mathrm{dom}(A) \mid Ax \in X\}$ and $A_X x := Ax$. It is obvious that A_X is closed (i.e. has a closed graph) if A is closed. (It should be noted that the X-realization of A is called by Kato "the part of A in X", e.g. [33, definition 4.2.1].)

If $A : \mathrm{dom}(A) \subset X \to X$ is a closed linear operator we denote by $\sigma(A)$ the spectrum (of the complexification of A in the case that $\mathbf{K} = \mathbf{R}$), by $\rho(A)$ the resolvent set, and by $R(\lambda, A) := (\lambda - A)^{-1}$ the resolvent of A at $\lambda \in \rho(A)$.

Let X be a real Banach space. A subset P of X is said to be a wedge if $P + P \subset P$, $\mathbf{R}_+ P \subset P$, $\bar{P} = P$, and $P \neq \emptyset$. A wedge P satisfying $P \cap (-P) = \{0\}$ is said to be a cone. Every wedge induces a preorder $\leqq$ (that is, a reflexive, transitive relation) in X by letting $x \leqq y$ iff $y - x \in P$. If P is a cone, then this preorder is an order (that is, it is also antisymmetric), and $X := (X, P)$ is said to be an ordered Banach space (OBS) with positive cone P. Clearly, $P = \{x \in X \mid x \geqq 0\}$, and we often write X^+ for the positive cone of the OBS X. Moreover we write $x > y$ if $x \geqq y$ but $x \neq y$.

If (X, P) and (Y, Q) are OBSs then $\mathscr{L}^+(X, Y) := \{T \in \mathscr{L}(X, Y) \mid T(P) \subset Q\}$ is a wedge in $\mathscr{L}(X, Y)$, the wedge of positive linear operators, inducing the natural preorder in $\mathscr{L}(X, Y)$. Thus $T \geqq 0$ means that $T \in \mathscr{L}^+(X, Y)$. In particular, X' is preordered by the dual wedge $P' := \mathscr{L}^+(X, \mathbf{R}) = \{x' \in X' \mid \langle x', x\rangle \geqq 0 \ \forall x \in P\}$. It is not difficult to see (on the basis of the basic separation theorems for convex sets) that $T \in \mathscr{L}^+(X)$ iff $T' \in \mathscr{L}^+(X')$. We refer to [27] for the elementary properties of OBSs which we shall use.

In this paper all function spaces are given the natural, that is, pointwise order and the corresponding positive cones are denoted by the superscript $+$. In general, all function spaces can be taken over $\mathbf{R}$ or $\mathbf{C}$. However, if we speak about order properties it is always understood that we consider real functions.

We write $A \in \mathscr{G}(X, M, \omega)$ if A is the infinitesimal generator of a strongly continuous semigroup $\{U(t) \mid t \geqq 0\}$ in $\mathscr{L}(X)$ such that

$$\|U(t)\| \leqq M e^{\omega t} \qquad \forall t \geqq 0.$$

Moreover, we let $\mathscr{G}(X) := \bigcup \{\mathscr{G}(X, M, \omega) \mid M \geqq 1, \omega \in \mathbf{R}\}$, and $e^{tA} := U(t)$. Thus A generates a strongly continuous contraction semigroup on X iff $A \in \mathscr{G}(X, 1, 0)$. We write $A \in \mathscr{H}(X)$ if A generates an analytic semigroup on X, where — in the case $\mathbf{K} = \mathbf{R}$ — complex analyticity refers to the complexification of X and the corresponding operators.

If $A \in \mathscr{G}(X)$ and $e^{tA} \in \mathscr{K}(X)$ for $t > 0$, then A is said to generate a compact semigroup. If X is an OBS then we write $A \in \mathscr{G}^+(X)$ if $e^{tA} \in \mathscr{L}^+(X)$ for $t > 0$, that is, if A generates a positive semigroup. Clearly, $\mathcal{G}^+(X, M, \omega) := \mathcal{G}(X, M, \omega) \cap \mathcal{G}^+(X)$ and $\mathcal{H}^+(X) = \mathcal{H}(X) \cap \mathcal{G}^+(X)$.

We constantly use the simple but important fact that, for each $\alpha \in \mathbf{R}$,

$$A \in \mathcal{G}(X, M, \omega) \qquad \text{iff } \alpha + A \in \mathcal{G}(X, M, \omega + \alpha)$$

and that

$$e^{t(\alpha + A)} = e^{\alpha t} e^{tA} \qquad \forall t > 0.$$

We refer to [7, 8, 14, 23] for the general theory of semigroups of linear operators.

We denote by $\nu := (\nu^1, \cdots, \nu^n)$ the outer normal on $\partial\Omega$. The outer conormal ν_a with respect to $\mathcal{A}$ is defined by $\nu_a^j := a_{jk}\nu^k$ for $j = 1, \cdots, n$. The norm in $L_p(\Omega)$ is denoted by $\|\cdot\|_p$, and $\|\cdot\|_{k,p}$ is the usual norm on the Sobolev space $W_p^k(\Omega)$. Finally, $\mathcal{D}(\Omega)$ is the space of all test functions on Ω.

If no confusion seems possible we denote by $\|\cdot\|_p$ also the norm on $\mathcal{L}(L_p(\Omega))$. Moreover, if $C \in \mathcal{L}(L_p(\Omega))$ such that $C \mid L_q(\Omega) \in \mathcal{L}(L_q(\Omega))$, where $1 \leqq p < q \leqq \infty$, we write simply $\|C\|_q$ for $\|C \mid L_q(\Omega)\|_q$.

2. Dual semigroups

Let $A \in \mathcal{G}(X, M, \omega)$. Then the *dual semigroup* $\{(e^{tA})' \mid t \in \mathbf{R}^+\}$ of $\{e^{tA} \mid t \in \mathbf{R}^+\}$ is a semigroup in $\mathcal{L}(X')$ satisfying

$$\|(e^{tA})'\| \leqq Me^{\omega t} \qquad \forall t \in \mathbf{R}^+.$$

But, in general, that is, if X is not reflexive, the dual semigroup is not strongly continuous. For this reason we let

$$X_A^\bullet := \{x' \in X' \mid [t \mapsto (e^{tA})'x'] \in C(\mathbf{R}^+, X')\},$$

that is, $X_A^\bullet$ is the largest subset of X' on which the dual semigroup is strongly continuous. It is not difficult to see that $X_A^\bullet$ is a closed linear subspace of X' which is invariant under $(e^{tA})'$, $t \geqq 0$, and satisfies

$$\operatorname{dom}(A') \subset X_A^\bullet$$

(e.g. [7, proposition 1.4.6]). Let

$$(e^{tA})^\bullet := (e^{tA})' \mid X_A^\bullet \qquad \forall t \in \mathbf{R}^+.$$

Then $\{(e^{tA})^\bullet \mid t \in \mathbf{R}^+\}$ is a strongly continuous semigroup in $\mathcal{L}(X_A^\bullet)$, the *strongly continuous* (restriction of the) *dual semigroup* of $\{e^{tA} \mid t \geqq 0\}$. We denote by $A^\bullet$ the infinitesimal generator of this strongly continuous dual semigroup so that

$$e^{tA^\bullet} = (e^{tA})^\bullet \qquad \forall t \in \mathbf{R}^+.$$

Then we have the following lemma due to Phillips [24] — as are all the results of this subsection.

(2.1) LEMMA. *$A^\bullet \in \mathcal{G}(X_A^\bullet, M, \omega)$ and $A^\bullet$ is the $X_A^\bullet$-realization of A'. Moreover, $X_A^\bullet = \mathrm{cl}_{X'}(\mathrm{dom}(A'))$, where $\mathrm{cl}_{X'}(\cdot)$ denotes the closure in X'. If X is reflexive, then $X_A^\bullet = X'$ and $A^\bullet = A'$.*

PROOF. [7, proposition 1.4.7] or [14, section 14.4]. □

By repeating the above construction with X and A replaced by $X_A^\bullet$ and $A^\bullet$, respectively, we obtain the space

$$X_A^{\bullet\bullet} := (X_A^\bullet)_{A^\bullet}^\bullet = \mathrm{cl}_{(X_A^\bullet)'}(\mathrm{dom}((A^\bullet)')).$$

In analogy with standard duality theory $X_A^\bullet$ and $X_A^{\bullet\bullet}$ are called the *A-dual* and *A-bidual of X*, respectively, expressing the fact that these spaces depend on A, in general.

In the remainder of this section we suppose that

$$A \in \mathcal{G}(X, 1, 0).$$

Then

$$\|x\| = \sup\{|\langle x^\bullet, x\rangle| \mid \|x^\bullet\| \leqq 1,\ x^\bullet \in X_A^\bullet\}$$

for each $x \in X$, and the map

$$\kappa_A^\bullet : X \to (X_A^\bullet)',$$

defined by

$$\langle \kappa_A^\bullet(x), x^\bullet\rangle := \langle x^\bullet, x\rangle \qquad \forall x^\bullet \in X_A^\bullet,$$

is a norm isomorphism onto a closed linear subspace of $X_A^{\bullet\bullet}$ (cf. [14, theorems 14.2.1 and 14.5 1]). Similarly as in standard duality theory *we identify* X with $\kappa_A^\bullet(X)$ by means of the norm isomorphism $\kappa_A^\bullet$ so that

$$X \subset X_A^{\bullet\bullet},$$

and X is said to be *A-reflexive* if $X = X_A^{\bullet\bullet}$.

In order to characterize A-reflexive spaces, A is said to have a *$\sigma(X, X_A^\bullet)$-compact resolvent* if, for some $\lambda \in \rho(A)$ — equivalently: for all $\lambda \in \rho(A)$ — the resolvent $R(\lambda, A)$ maps bounded sets into $\sigma(X, X_A^\bullet)$-compact sets, that is, into sets which are compact with respect to the $X_A^\bullet$-topology of X.

(2.2) THEOREM. *X is A-reflexive iff A has a $\sigma(X, X_A^\bullet)$-compact resolvent. The* *latter is the case if A has a weakly compact resolvent, thus, in particular, if either X is reflexive or A has a compact resolvent.*

PROOF. [14, section 14.6]. □

3. Semigroup duality

In this section we extend slightly Phillips' theory of strongly continuous dual semigroups by considering semigroups in $X_A^\bullet$ whose generators are distinct from $A^\bullet$ and do not necessarily commute with $A^\bullet$.

Throughout this section we suppose that $A \in \mathscr{G}(X, 1, 0)$ *and that*

$$B : \mathrm{dom}(B) \subset X \to X$$

is a densely defined linear operator.

We define the *A-dual $B_A^\bullet$ of B* to be the $X_A^\bullet$-realization of B'. Thus $B_A^\bullet$ is a linear operator in $X_A^\bullet$. If $B_A^\bullet$ is also densely defined then we define the *A-bidual $B_A^{\bullet\bullet}$ of B* by

$$B_A^{\bullet\bullet} := (B_A^\bullet)_{A^\bullet}^\bullet,$$

that is,

$$\mathrm{dom}(B_A^{\bullet\bullet}) = \{x^{\bullet\bullet} \in \mathrm{dom}((B_A^\bullet)') \cap X_A^{\bullet\bullet} \mid (B_A^\bullet)' x^{\bullet\bullet} \in X_A^{\bullet\bullet}\}$$

and $B_A^{\bullet\bullet} x^{\bullet\bullet} = (B_A^\bullet)' x^{\bullet\bullet}$.

The following lemmas exhibit a strong analogy to standard duality theory.

(3.1) LEMMA. *$B_A^\bullet$ is closed. If $B_A^\bullet$ is densely defined, then $B_A^{\bullet\bullet} \supset B$.*

PROOF. The closedness follows from the closedness of B'.

Let $B_A^\bullet$ be densely defined so that $B_A^{\bullet\bullet}$ exists, and let $x \in \mathrm{dom}(B)$. Then

$$\langle \kappa_A^\bullet(Bx), x^\bullet \rangle = \langle x^\bullet, Bx \rangle = \langle B_A^\bullet x^\bullet, x \rangle = \langle \kappa_A^\bullet(x), B_A^\bullet x^\bullet \rangle$$

for all $x^\bullet \in \mathrm{dom}(B_A^\bullet)$. Hence $\kappa_A^\bullet(x) \in \mathrm{dom}((B_A^\bullet)')$ and $(B_A^\bullet)' \kappa_A^\bullet(x) = \kappa_A^\bullet(Bx) \in X_A^{\bullet\bullet}$. Thus $x \equiv \kappa_A^\bullet(x) \in \mathrm{dom}(B_A^{\bullet\bullet})$ and $B_A^{\bullet\bullet} x = \kappa_A^\bullet(Bx) \equiv Bx$, that is, $B_A^{\bullet\bullet} \supset B$. □

(3.2) LEMMA. *Let B be closed and suppose that $R(\lambda, B')(X_A^\bullet) \subset X_A^\bullet$ for all $\lambda \in \rho(B)$. Then $\rho(B) \subset \rho(B_A^\bullet)$ and $R(\lambda, B_A^\bullet) = R(\lambda, B') \mid X_A^\bullet$ for all $\lambda \in \rho(B)$.*

PROOF. Let $\lambda \in \rho(B)$ and $x^\bullet \in \mathrm{dom}(B_A^\bullet)$. Then, since $\rho(B) = \rho(B')$ and $R(\lambda, B') = R(\lambda, B)'$,

$$R(\lambda, B')(\lambda - B_A^\bullet)x^\bullet = R(\lambda, B')(\lambda - B')x^\bullet = x^\bullet = (\lambda - B')R(\lambda, B')x^\bullet. \tag{3.1}$$

Thus $R(\lambda, B')(\mathrm{dom}(B^\bullet_A)) \subset \mathrm{dom}(B^\bullet_A)$. Hence, if $y^\bullet \in X^\bullet_A$ is arbitrary,

$$y^\bullet = (\lambda - B')R(\lambda, B')y^\bullet = (\lambda - B^\bullet_A)R(\lambda, B')y^\bullet. \tag{3.2}$$

Now it follows from (3.1) and (3.2) that $\lambda \in \rho(B^\bullet_A)$ and $R(\lambda, B')X^\bullet_A = R(\lambda, B^\bullet_A)$.

□

Observe that, trivially, $R(\lambda, B')(X^\bullet_A) \subset X^\bullet_A$ for all $\lambda \in \rho(B)$ if $\mathrm{dom}(B') \subset X^\bullet_A$.

Suppose now that $B \in \mathcal{G}(X)$. Then, by Phillips' theory of Section 2, $B^\bullet \in \mathcal{G}(X^\bullet_B)$. However, in practical cases the space $X^\bullet_A$ may be easy to determine but not the space $X^\bullet_B$. For these reasons we are interested in situations where $B^\bullet_A$ generates a semigroup on $X^\bullet_A$.

(3.3) THEOREM. *Let $B \in \mathcal{G}(X, M, \omega)$ and suppose that $B^\bullet_A$ is densely defined and $R(\lambda, B')(X^\bullet_A) \subset X^\bullet_A$ for all $\lambda \in \rho(B)$. Then $B^\bullet_A \in \mathcal{G}(X^\bullet_A, M, \omega)$ and $e^{tB^\bullet_A} = (e^{tB})' \mid X^\bullet_A$ for $t \geqq 0$.*

PROOF. Since $B \in \mathcal{G}(X, M, \omega)$, it is closed, densely defined, $(\omega, \infty) \subset \rho(B)$, and — by the necessity part of the general Hille–Yosida theorem (e.g. [14, theorem 12.3.1]) —

$$\|R(\lambda, B)^n\| \leqq M(\lambda - \omega)^{-n} \qquad \forall \lambda > \omega, \quad n \in \mathbf{N}.$$

Thus $B^\bullet_A$ is closed by Lemma (3.1) and densely defined by assumption. Moreover Lemma (3.2) implies $(\omega, \infty) \subset \rho(B^\bullet_A)$ and

$$\|R(\lambda, B^\bullet_A)^n\| \leqq \|R(\lambda, B')^n\| = \|R(\lambda, B)^n\| \leqq M(\lambda - \omega)^{-n}$$

for $\lambda > \omega$ and $n \in \mathbf{N}$. Thus $B^\bullet_A \in \mathcal{G}(X^\bullet_A, M, \omega)$ by the sufficiency part of the Hille–Yosida theorem.

Recall (e.g. [14, theorem 11.6.6]) that

$$e^{tB}x = \lim_{n \to \infty} \left[\frac{n}{t} R\left(\frac{n}{t}, B\right)\right]^n x = \lim_{n \to \infty} \left(1 - \frac{t}{n} B\right)^{-n} x \tag{3.3}$$

for each $x \in X$ and every $t \geqq 0$. Hence, again by Lemma (3.2),

$$\langle x^\bullet, e^{tB}x\rangle = \lim_{n \to \infty} \left\langle x^\bullet, \left[\frac{n}{t} R\left(\frac{n}{t}, B\right)\right]^n x\right\rangle$$

$$= \lim_{n \to \infty} \left\langle \left[\frac{n}{t} R\left(\frac{n}{t}, B'\right)\right]^n x^\bullet, x\right\rangle = \lim_{n \to \infty} \left\langle \left[\frac{n}{t} R\left(\frac{n}{t}, B^\bullet_A\right)\right]^n x^\bullet, x\right\rangle = \langle e^{tB^\bullet_A} x^\bullet, x\rangle$$

for all $x^\bullet \in X^\bullet_A$ and $x \in X$. This proves the last assertion. □

(3.4) COROLLARY. *Let $B \in \mathcal{G}(X)$, let $R(\lambda, B')(X^\bullet_A) \subset X^\bullet_A$ for all $\lambda \in \rho(B)$, and let $B^\bullet_A$ be densely defined. Then $X^\bullet_A \subset X^\bullet_B$ and $e^{tB^\bullet} \mid X^\bullet_A = e^{tB^\bullet_A}$ for $t \geqq 0$.*

PROOF. This follows from Theorem (3.3), the fact that $X_B^\bullet$ is the largest subspace of X' on which the dual semigroup $\{(e^{tB})' \mid t \geqq 0\}$ is strongly continuous, and from $(e^{tB})' \mid X_B^\bullet = e^{tB^\bullet}$. □

It is clear that $X_B^\bullet = X'$ if $B \in \mathscr{L}(X)$. This shows that, in general, the inclusion $X_A^\bullet \subset X_B^\bullet$ is proper.

(3.5) THEOREM. *Let $B \in \mathscr{G}(X)$, let $R(\lambda, B')(X_A^\bullet) \subset X_A^\bullet$ for all $\lambda \in \rho(B)$, and let $B_A^\bullet$ be densely defined. Then*

(i) $B \in \mathscr{H}(X) \Rightarrow B_A^\bullet \in \mathscr{H}(X_A^\bullet)$;

(ii) $B \in \mathscr{G}^+(X) \Rightarrow B_A^\bullet \in \mathscr{G}^+(X_A^\bullet)$;

(iii) $e^{tB} \in \mathscr{K}(X) \Rightarrow e^{tB_A^\bullet} \in \mathscr{K}(X_A^\bullet)$.

PROOF. (i) It is well known that $C \in \mathscr{H}(X)$ iff C is densely defined and closed and there exist constants $c \in \mathbf{R}^+$ and $\gamma \in \mathbf{R}$ such that $\rho(C) \supset \{\lambda \in \mathbf{C} \mid \operatorname{Re} \lambda \geqq \gamma\}$ and

$$\|R(\lambda, C)\| \leqq c(1 + |\lambda|)^{-1} \qquad \text{for } \operatorname{Re} \lambda \geqq \gamma$$

(e.g. [16, theorem 13.2]). Thus, since $\|R(\lambda, A')\| = \|R(\lambda, A)\|$, the assertion follows from Lemmas (3.1) and (3.2).

(ii) Since $e^{tB} \geqq 0$ implies $(e^{tB})' \geqq 0$, the assertion follows from Theorem (3.3).

(iii) is again a consequence of Theorem (3.3) and Schauder's theorem on the dual of a compact linear operator (e.g. [37, theorem X.4]). □

(3.6) PROPOSITION. *Let X be A-reflexive, let B be closed with nonempty resolvent set, assume that $R(\lambda, B')(X_A^\bullet) \subset X_A^\bullet$ for all $\lambda \in \rho(B)$, and let $B_A^\bullet$ be densely defined. Then $B_A^{\bullet\bullet} = B$.*

PROOF. Let $\lambda \in \rho(B)$. Then $(\lambda - B)_A^\bullet = \lambda - B_A^\bullet$ and $(\lambda - B)_A^{\bullet\bullet} = \lambda - B_A^{\bullet\bullet} \supset \lambda - B$ by Lemma (3.1). Moreover, by Lemma (3.2),

$$\rho(B) \subset \rho(B_A^\bullet) \subset \rho(B_A^{\bullet\bullet}).$$

Hence $\lambda - B_A^{\bullet\bullet}$ and $\lambda - B$ are both bijective which shows that $\lambda - B_A^{\bullet\bullet}$ cannot be a proper extension of $\lambda - B$. Thus $B_A^{\bullet\bullet} = B$. □

4. Formally adjoint boundary value problems

We define the *formally adjoint differential operator* $\mathscr{A}^\#$ of $\mathscr{A}$ by

$$\mathscr{A}^\# v := -D_j(a_{jk} D_k v) - D_j(a_j v) + a_0 v.$$

Then, by the divergence theorem,

$$(4.1)\qquad (\mathscr{A}u \mid v)_{L_2} - (u \mid \mathscr{A}^{\#} v)_{L_2} = -\int_{\Gamma_1} \left(\frac{\partial u}{\partial \nu_a} v - \frac{\partial v}{\partial \nu_a} u - a_j \nu^j uv \right) d\sigma$$

for all $u, v \in C_0^2(\bar{\Omega}) := \{u \in C^2(\bar{\Omega}) \mid u \mid \Gamma_0 = 0\}$, provided $a_j \in C^1(\bar{\Omega})$, $j = 1, \dots, n$.

We let

$$\rho := (\nu_a \mid \nu)/(\beta \mid \nu) \in C^1(\Gamma_1),$$

where $(\cdot \mid \cdot)$ denotes the Euclidian inner product in $\mathbf{R}^n$, and define a tangential vector field on Γ_1 of class C^1 by

$$\tau := \nu_a - \rho\beta,$$

so that

$$(4.2)\qquad \frac{\partial}{\partial \nu_a} = \rho \frac{\partial}{\partial \beta} + \frac{\partial}{\partial \tau}\,.$$

Since $\rho(x) > 0$ for all $x \in \Gamma_1$, we can define $\beta^{\#} \in C^1(\Gamma_1, \mathbf{R}^n)$ by

$$\rho\beta^{\#} := \nu_a + \tau$$

so that

$$(4.3)\qquad \frac{\partial}{\partial \nu_a} = \rho \frac{\partial}{\partial \beta^{\#}} - \frac{\partial}{\partial \tau}\,.$$

Since $\rho(\beta^{\#} \mid \nu) = (\nu_a \mid \nu)$, we see that $\beta^{\#}$ is an outward pointing, nowhere tangent vector field on Γ_1.

It follows from (4.2) and (4.3) that

$$\frac{\partial u}{\partial \nu_a} v - \frac{\partial v}{\partial \nu_a} u = \rho \left(\frac{\partial u}{\partial \beta} v - \frac{\partial v}{\partial \beta^{\#}} u \right) + \frac{\partial (uv)}{\partial \tau}\,.$$

Since, for $u, v \in C^1(\bar{\Omega})$,

$$\frac{\partial (uv)}{\partial \tau} = (\operatorname{grad}(uv) \mid \tau) = (\operatorname{grad}_{\Gamma_1}(uv) \mid \tau)_{\Gamma_1},$$

where $(\cdot \mid \cdot)_{\Gamma_1}$ denotes the inner product (that is, Riemannian metric) on Γ_1 induced by $(\cdot \mid \cdot)$ and $\operatorname{grad}_{\Gamma_1}(uv)$ is the corresponding gradient of $(uv) \mid \Gamma_1$, it follows that

$$\frac{\partial (uv)}{\partial \tau} = \operatorname{div}_{\Gamma_1}(uv\tau) - uv \operatorname{div}_{\Gamma_1}(\tau),$$

where $\operatorname{div}_{\Gamma_1}$ is the divergence on Γ_1.

Finally we define $\beta_0^\# \in C(\Gamma_1)$ by

$$\rho\beta_0^\# := \rho\beta_0 + a_j\nu^j + \operatorname{div}_{\Gamma_1}(\tau) \tag{4.4}$$

and the *formally adjoint boundary operator* $\mathscr{B}^\#$ of $\mathscr{B}$ by

$$\mathscr{B}^\# v := \begin{cases} v & \text{on } \Gamma_0, \\ \dfrac{\partial v}{\partial \beta^\#} + \beta_0^\# v & \text{on } \Gamma_1. \end{cases}$$

Then, by applying the divergence theorem on Γ_1 to the right hand side of (4.1), we obtain *Green's formula*

$$(\mathscr{A}u \mid v)_{L_2} - (u \mid \mathscr{A}^\# v)_{L_2} = -\int_{\Gamma_1} \rho(v\mathscr{B}u - u\mathscr{B}^\# v)d\sigma$$

for all $u, v \in C_0^2(\bar{\Omega})$.

We can write $\mathscr{A}^\#$ in the form

$$\mathscr{A}^\# v = -D_j(a_{jk}^\# D_k v) + a_j^\# D_j v + a_0^\# v,$$

where

$$a_{jk}^\# := a_{jk}, \quad a_j^\# := -a_j, \quad a_0^\# := a_0 - D_j a_j. \tag{4.5}$$

Then we see from (4.4) and (4.5) that

$(\mathscr{A}, \mathscr{B})$ and $(\mathscr{A}^\#, \mathscr{B}^\#)$ are regular elliptic BVPs provided

$$a_j, a_{jk} \in C^1(\bar{\Omega}), \qquad a_0 \in C(\bar{\Omega})$$

and

$$\Gamma_1 \in C^3 \qquad \text{if } \beta \neq \nu_a.$$

Of course, these conditions are sufficient but not necessary.

If we repeat the above procedure, starting with $(\mathscr{A}^\#, \mathscr{B}^\#)$ instead of $(\mathscr{A}, \mathscr{B})$, we see that $\rho^\# = \rho$ and $\tau^\# = -\tau$, which, together with (4.5), implies

$$(\mathscr{A}^\#)^\# = \mathscr{A} \quad \text{and} \quad (\mathscr{B}^\#)^\# = \mathscr{B}.$$

Moreover, the BVP $(\mathscr{A}, \mathscr{B})$ is *formally self-adjoint*, that is, $\mathscr{A}^\# = \mathscr{A}$ and $\mathscr{B}^\# = \mathscr{B}$ iff $a_j = 0$ and $\beta = \alpha\nu_a$ for some $\alpha \in C^1(\Gamma_1)$ satisfying $\alpha(x) > 0$ for all $x \in \Gamma_1$. In this case $\tau = 0$ and $\rho = 1/\alpha$.

It should be noted that the above results are also true and meaningful if $n = 1$, that is, if $\Omega := (x_0, x_1)$ is a bounded open interval in $\mathbf{R}$, provided $\nu(x_0) := -1$, $\nu(x_1) := 1$, and σ is the counting measure on $\partial\Omega$.

Of course it is well known that there are Green formulas for elliptic BVPs (e.g. [20, section I.6] or, in much greater generality, [19, section II.2]). It is the purpose of the above considerations to give a normalization of the formally adjoint boundary operator $\mathcal{B}^{\#}$ (through the weight function ρ), a closed representation of $\mathcal{B}^{\#}$, and to exhibit clearly the regularity properties of the coefficients of $\mathcal{B}^{\#}$.

5. Trace and density theorems

Let $1<p<\infty$ and $0 \leqq s \leqq 2$. Then $W_p^s(M)$ denotes the standard Sobolev (–Slobodeckii) space on M, where M equals Ω or Γ_i, $i=0,1$ (e.g. [1, 7, 21, 34]).

We denote by

$$\gamma_i \in \mathcal{L}(C^2(\bar{\Omega}), C^2(\Gamma_i)), \qquad i=0,1,$$

the trace operators, $\gamma_i(u):=u \mid \Gamma_i$, as well as their continuous extensions

$$\gamma_i \in \mathcal{L}(W_p^k(\Omega), W_p^{k-1/p}(\Gamma_i)), \qquad k=1,2, \quad i=0,1$$

(e.g. [1, 7, 21, 34]). By means of γ_i we can rewrite the boundary operator $\mathcal{B}$ more precisely in the form

$$\mathcal{B}=(\gamma_0, \beta^j \gamma_1 \circ D_j + \beta_0 \gamma_1),$$

which shows that

$$\mathcal{B} \in \mathcal{L}(W_p^2(\Omega), W_p^{2-1/p}(\Gamma_0) \times W_p^{1-1/p}(\Gamma_1)).$$

The following important extension lemma, generalizing [2, lemma 3.2], shows that $\mathcal{B}$ possesses a continuous right inverse enjoying some additional properties.

(5.1) LEMMA. *There exists*

$$\mathcal{R} \in \mathcal{L}(W_p^{2-1/p}(\Gamma_0) \times W_p^{1-1/p}(\Gamma_1), W_p^2(\Omega))$$

satisfying

$$\mathcal{B}\mathcal{R}=\mathrm{id} \quad \text{and} \quad \gamma_1 \circ \mathcal{R}=0$$

as well as

$$\mathcal{R}(C^2(\Gamma_0) \times C^1(\Gamma_1)) \subset C^2(\bar{\Omega}).$$

PROOF. By means of local coordinates and standard partition of unity arguments we can reduce the problem to the case where Ω is replaced by the upper halfspace $\mathbf{H}^n := \{x \in \mathbf{R}^n \mid x^n > 0\}$ and all functions involved have compact supports and are at most of class C^2 (compare, for example, the proof of lemma (3.2) in [2]). If $\partial \mathbf{H}^n = \mathbf{R}^{n-1} \times \{0\} \equiv \mathbf{R}^{n-1}$ corresponds to (a portion of) Γ_0, the assertion is simply the standard (inverse) trace theorem (e.g. [21, theorem II.5.8]). Thus we can assume that $\partial \mathbf{H}^n$ corresponds to (a portion of) Γ_1.

Let E be an arbitrary Banach space and let $U(t)$, $V(t)$, $t \geqq 0$, be strongly continuous semigroups on E. Then we define the convolution $U * V$ by

$$U * V(t)x := \int_0^t U(t-\tau)V(\tau)x d\tau \qquad \forall t \geqq 0, \quad x \in E.$$

Moreover, we let $U^{((2))} := U * U$.

Let $m := n-1$ and, for $j = 1, \cdots, m$, let $U_j := \{U_j(t) \mid t \geqq 0\}$ be the translation semigroup

$$U_j(t)u(y) := u(y^1, \cdots, y^{j-1}, y^j + t, y^{j+1}, \cdots, y^m).$$

Then it is well known that the U_j are strongly continuous, pairwise commuting, positive semigroups on E, where $E := L_p(\mathbf{R}^m)$ or $E := \mathrm{BUC}(\mathbf{R}^m)$, the space of bounded and uniformly continuous functions on $\mathbf{R}^m$. Moreover, the infinitesimal generator A_j of U_j is given by

$$\mathrm{dom}(A_j) = \{u \in E \mid D_j u \in E\} \quad \text{and} \quad A_j u = D_j u$$

(e.g. [7, sections 1.3.3 and 4.3.1]).

Let $V(t) := t\, \mathrm{id}_E$ for $t \geqq 0$ and define $\mathcal{R}u$ by

$$\mathcal{R}u(t) := \varphi(t) t^{-2m} V * U_1^{((2))} * \cdots * U_m^{((2))}(t)u$$

for $t > 0$ and $u \in E$, where $\varphi \in C^2(\mathbf{R}^+, \mathbf{R})$ has compact support and satisfies $\varphi(0) > 0$ and $D\varphi(0) = 0$. If $\varphi(0)$ is suitably chosen and if $u \in \bigcap_{j=1}^m \mathrm{dom}(A_j)$, it follows from the very general results of J. L. Lions [17] that $\mathcal{R}u \in C^2(\mathbf{R}^+, E)$, that $\mathcal{R}u(0) = 0$ and $D\mathcal{R}u(0) = u$, and that $\mathcal{R} \in \mathcal{L}(T_1, T_2)$, where T_1 and T_2 are appropriate trace (that is, special interpolation) spaces. Choosing in particular $E = L_p(\Omega)$, it follows that, up to equivalent norms, $T_1 = W_p^{1-1/p}(\partial \mathbf{H}^n)$ and $T_2 = W_p^2(\mathbf{H}^n)$ (cf. in particular [17, section 9]). From these facts the assertion is readily deduced. □

It should be remarked that a more direct but much more technical proof of Lemma (5.1) could be based on the results in [21] (cf. in particular [21, lemma II.5.8]). In [21] the explicit use of semigroups and interpolation spaces is avoided, but the principal ideas of [21] and [17] are the same.

As an easy consequence of Lemma (5.1) we obtain the following approximation lemma. Here and in the following we use the following notation: if $X(\Omega)$ is a space of functions on Ω, we let

$$X_{\mathscr{B}}(\Omega):=\{u \in X(\Omega) \mid \mathscr{B}u=0\}=X(\Omega)\cap \ker(\mathscr{B}),$$

whenever $\mathscr{B}$ is well defined on $X(\Omega)$. Similarly,

$$X_0(\Omega):=\{u \in X(\Omega) \mid \gamma_0(u)=0\}=X(\Omega)\cap \ker(\gamma_0).$$

Observe that there is no restriction on $u \mid \Gamma_1$ if $u \in X_0(\Omega)$.

(5.2) LEMMA. *$C^2_{\mathscr{B}}(\bar{\Omega})$ is dense in $W^2_{p,\mathscr{B}}(\Omega)$.*

PROOF. Let $u \in W^2_{p,\mathscr{B}}(\Omega)$ and $\varepsilon>0$ be arbitrary. Since $C^2(\bar{\Omega})$ is dense in $W^2_p(\Omega)$, there is a $v \in C^2(\bar{\Omega})$ satisfying $\|u-v\|_{2,p}<\varepsilon(1+\|\mathscr{R}\mathscr{B}\|)^{-1}$. Then $\mathscr{B}v \in C^2(\Gamma_0)\times C^1(\Gamma_1)$ and $w:=\mathscr{R}\mathscr{B}v \in C^2(\bar{\Omega})$ by Lemma (5.1). Thus $v-w \in C^2_{\mathscr{B}}(\bar{\Omega})$ and

$$\|u-(v-w)\|_{2,p} \leqq \|u-v\|_{2,p}+\|w\|_{2,p}$$
$$=\|u-v\|_{2,p}+\|\mathscr{R}\mathscr{B}(u-v)\|_{2,p} \leqq (1+\|\mathscr{R}\mathscr{B}\|)\|u-v\|_{2,p}<\varepsilon,$$

which proves the assertion. □

In the following we let

$$C^2_c(\Omega\cup\Gamma_1):=\{u \in C^2(\bar{\Omega}) \mid \operatorname{supp}(u)\subset \Omega\cup\Gamma_1\}$$

and prove another density result which will be useful later, namely

(5.3) LEMMA. *$C^2_{c,\mathscr{B}}(\Omega\cup\Gamma_1)$ is dense in $C_0(\bar{\Omega})$.*

PROOF. By means of local coordinates and a standard "translation argument" based on the fact that translation is strongly continuous in $\mathrm{BUC}(\mathbf{R}^n)$, it follows easily that $C^2_c(\Omega\cup\Gamma_1)$ is dense in $C_0(\bar{\Omega})$ (cf. the proof of [1, theorem 3.18]).

Let $u \in C_0(\bar{\Omega})$ and $\varepsilon>0$ be given. Then there is a $v \in C^2_c(\Omega\cup\Gamma_1)$ satisfying $\|u-v\|_\infty<\varepsilon/2$. Let $w:=\mathscr{R}\mathscr{B}v$ and, for each $\delta>0$, let $\varphi_\delta \in \mathscr{D}(\mathbf{R}^n)$ satisfy $0\leqq\varphi_\delta\leqq 1$, $\operatorname{supp}(\varphi_\delta)\in \mathbf{B}(\Gamma_1,\delta)$, and $\varphi_\delta \mid \mathbf{B}(\Gamma_1,\delta/2)=1$, where $\mathbf{B}(\Gamma_1,\rho):=\{x\in\mathbf{R}^n \mid \operatorname{dist}(x,\Gamma_1)<\rho\}$. Then $\varphi_\delta w \in C^2_c(\Omega\cup\Gamma_1)$ for δ sufficiently small and $\|\varphi_\delta w\|_\infty \to 0$ as $\delta\to 0$, since $\varphi_\delta w \mid \Gamma_1=0$ by Lemma (5.1). Hence we find a $\delta>0$ such that $z:=\varphi_\delta w \in C^2_c(\Omega\cup\Gamma_1)$ and $\|z\|_\infty<\varepsilon/2$. Since $\mathscr{B}z=\mathscr{B}w=\mathscr{B}v$, it follows that $v-z \in C^2_{c,\mathscr{B}}(\Omega\cup\Gamma_1)$ and

$$\|u-(v-z)\|_\infty \leqq \|u-v\|_\infty+\|z\|_\infty<\varepsilon,$$

which proves the assertion. □

6. The maximum principle

In this section we give an extension of the well known "inverse positivity" result for elliptic BVPs of second order, which are obtained from Hopf's maximum principle. Since *we do not presuppose any sign condition for* β_0, our results seem to be new even in the case of classical solutions.

In the following we denote by $\tilde{\nu} : \Gamma_0 \to \mathbf{R}^n$ an arbitrary outward pointing, nowhere tangent vector field.

(6.1) THEOREM. *There exists a constant* $\hat{\lambda} \in \mathbf{R}$ *such that*

$$(6.1) \qquad u \in W_p^2(\Omega), \quad p > n, \quad (\mathcal{A} + \lambda)u \geqq 0, \quad \mathcal{B}u \geqq 0 \quad \text{and} \quad \lambda > \hat{\lambda}$$

imply $u \geqq 0$. *Moreover, if* $u \neq 0$ *then* $u(x) > 0$ *for all* $x \in \Omega \cup \Gamma_1$ *and* $(\partial u / \partial \tilde{\nu})(x) < 0$ *for* $x \in \Gamma_0$.

PROOF. Let $v := (0, \|\beta_0\|_\infty) \in C^2(\Gamma_0) \times C^1(\Gamma_1)$, and let $\varphi \in C^2(\mathbf{R}^n)$ satisfy $0 \leqq \varphi \leqq 1$, $\text{supp}(\varphi) \subset \mathbf{B}(\Gamma_1, 2\varepsilon)$, and $\varphi \mid \mathbf{B}(\Gamma_1, \varepsilon) = 1$, where $\varepsilon > 0$ is so small that φ vanishes in a neighborhood of Γ_0 and $w := 1 + \varphi \mathcal{R} v \geqq \frac{1}{2}$. Since $\mathcal{R} v \in C^2(\bar{\Omega})$ and $\mathcal{R} v \mid \Gamma_1 = 0$, this choice of ε is possible. Then $w \in C^2(\bar{\Omega})$, $w = 1$ in a neighborhood of Γ_0, and $\mathcal{B} w \geqq 0$ by Lemma (5.1). Moreover, $(\mathcal{A} + \lambda)w > 0$ for $\lambda > 2\|\mathcal{A} w\|_\infty =: \hat{\lambda}$.

Let (6.1) be satisfied. Then $u \in C^1(\bar{\Omega})$ by the Sobolev imbedding theorem, and u is a.e. twice classically differentiable [30, theorem VIII.1]. (Clearly, we identify the equivalence class $u \in W_p^2(\Omega)$ with an appropriate representation, as is usual in the statement of imbedding theorems.) Now the assertion follows by an obvious combination of the generalized maximum principle of Protter and Weinberger [25, section II.5] with Bony's maximum principle [5]. □

It should be observed that the above theorem remains true if the coefficients of $\mathcal{A}$ are only supposed to belong to $L_\infty(\Omega)$.

7. Elliptic boundary value problems in $L_p(\Omega)$, $1 < p < \infty$

For each $p \in (1, \infty)$ the L_p*-realization* A_p of the BVP $(\mathcal{A}, \mathcal{B})$ is defined by

$$A_p : \text{dom}(A_p) \subset L_p(\Omega) \to L_p(\Omega),$$

where $\text{dom}(A_p) := W^2_{p,\mathcal{B}}(\Omega)$ and $A_p u := \mathcal{A} u$. Then it is well known (e.g. [33, section 3.8]) that

$$-A_p \in \mathcal{H}(L_p(\Omega)).$$

Standard regularity theory implies that the spectrum of A_p is independent of p and that

$$(7.1)\qquad (\lambda + A_p)^{-1} \,|\, L_q(\Omega) = (\lambda + A_q)^{-1}$$

for $\lambda \in \rho(-A_p)$ and $1 < p \leqq q < \infty$. Hence (7.1) and the representation formula (3.3) show that e^{-tA_p} leaves $L_q(\Omega)$ invariant and that

$$(7.2)\qquad e^{-tA_p} \,|\, L_q(\Omega) = e^{-tA_q} \qquad \text{for } 1 < p \leqq q < \infty.$$

Since $W^2_{p,\mathscr{B}}(\Omega) \hookrightarrow L_p(\Omega)$, A_p has a compact resolvent. Hence it follows from Pazy's theorem [22, theorem 3.3] that $e^{-tA_p} \in \mathscr{K}(L_p(\Omega))$ for $t > 0$.

Let $u \in C^+(\bar{\Omega})$. Then (7.1) and Theorem (6.1) imply that $(\lambda + A_p)^{-1} u \geqq 0$ for all $\lambda > \hat{\lambda}$. Since $C^+(\bar{\Omega})$ is dense in $L_p^+(\Omega)$, it follows that $(\lambda + A_p)^{-1} \geqq 0$ for $\lambda > \hat{\lambda}$. From this we deduce that $e^{-tA_p} \geqq 0$ for $t \geqq 0$ (cf. [14, theorem 11.7.2]). In summary:

$-A_p$ generates a compact, positive, analytic semigroup on $L_p(\Omega)$.

In the following A'_p denotes the dual of A_p and $A^\#_p$ the L_p-realization of the formally adıoint BVP $(\mathscr{A}^\#, \mathscr{B}^\#)$. For completeness we include a simple proof of the following well-known

(7.1) THEOREM. *Suppose that $(\mathscr{A}^\#, \mathscr{B}^\#)$ is a regular elliptic BVP. Thus $A'_p = A^\#_{p'}$, where $p' := p/(p-1)$.*

PROOF. Green's formula and Lemma (5.2) imply

$$\langle v, A_p u \rangle = \langle A^\#_{p'} v, u \rangle \qquad \forall u \in \mathrm{dom}(A_p), \quad v \in \mathrm{dom}(A^\#_{p'}).$$

Thus $A'_p \supset A^\#_{p'}$. Since $\lambda + A'_p = (\lambda + A_p)'$ and $\lambda + A^\#_{p'}$ are both bijective for sufficiently large $\lambda \in \mathbf{R}$, we see that A'_p cannot be a proper extension of $A^\#_{p'}$. □

8. Boundary value problems and semigroups in $C(\bar{\Omega})$

We define the *L_∞-realization* A_∞ of $(\mathscr{A}, \mathscr{B})$ by

$$\mathrm{dom}(A_\infty) := \{u \in W^2_{p,\mathscr{B}}(\Omega) \,|\, A_p u \in L_\infty(\Omega)\}$$

and $A_\infty u := A_p u$, where $p \in (1, \infty)$ is arbitrary. It follows from (7.1) and the Sobolev imbedding theorem that A_∞ is independent of p and that A_∞ is the $L_\infty(\Omega)$-realization of A_p (in the sense of the definition of Section 1). Clearly A_∞ is a closed linear operator in $L_\infty(\Omega)$.

The following lemma shows that A_∞ is not densely defined and that the closure of its domain is independent of the special operator.

(8.1) LEMMA. $\overline{\mathrm{dom}(A_\infty)} = C_0(\bar{\Omega})$.

PROOF. The assertion follows from the obvious inclusions

$$C^2_{c,\mathcal{B}}(\Omega \cup \Gamma_1) \subset \mathrm{dom}(A_\infty) \subset W^2_{p,\mathcal{B}}(\Omega) \subset C_0(\bar{\Omega}),$$

where $p > n$, and from Lemma (5.3). □

Finally we define the *C-realization* A of $(\mathcal{A}, \mathcal{B})$ to be the $C_0(\bar{\Omega})$-realization of A_∞. Thus

$$\mathrm{dom}(A) := \{u \in \mathrm{dom}(A_\infty) \mid A_\infty u \in C_0(\bar{\Omega})\}$$

and $Au := A_\infty u$. Observe that $C^2_{c,\mathcal{B}}(\Omega \cup \Gamma_1) \subset \mathrm{dom}(A)$. Hence A is densely defined by Lemma (5.3).

(8.2) THEOREM. *$-A$ generates a compact, positive, analytic semigroup on $C_0(\bar{\Omega})$.*

PROOF. The fact that $-A \in \mathcal{H}(C_0(\bar{\Omega}))$ follows from the much more general results of Stewart [30, 31]. The Rellich–Kondrachov theorem implies the compactness of the resolvent of A. Thus, again by Pazy's theorem, $-A$ generates a compact semigroup. The positivity assertion follows also again from Theorem (6.1). □

The following theorem is essentially known.

(8.3) THEOREM. *If $a_0 \geqq 0$ and $\beta_0 \geqq 0$ then $-A \in \mathcal{G}(C_0(\bar{\Omega}), 1, 0)$.*

PROOF. Let $\lambda > 0$ and $u \in \mathrm{dom}(A)$, and let $v := \|(\lambda + A)u\|_\infty / \lambda$. Then

$$(\mathcal{A} + \lambda)(v \pm u) \geqq 0, \qquad \mathcal{B}(v \pm u) \geqq 0$$

so that $-v \leqq u \leqq v$ by the maximum principle (cf. the proof of Theorem (6.1)). Thus $\lambda \|u\|_\infty \leqq \|(\lambda + A)u\|_\infty$ for all $\lambda > 0$ and $u \in \mathrm{dom}(A)$, and the assertion follows from the Hille–Yosida theorem. □

(8.4) REMARK. If $-A \in \mathcal{G}(C_0(\bar{\Omega}), 1, 0)$, then $a_0 \geqq 0$ by a result of Sinestrari [28]. If $n = 1$, then Fattorini [10] has shown that also $\beta_0 \geqq 0$. By the same arguments it should be possible to show that $\beta_0 \geqq 0$ in the general case. We leave the details to the reader. □

It is easily verified that $\sigma(A) = \sigma(A_p)$, $1 < p < \infty$.

9. Elliptic boundary value problems in $L_1(\Omega)$

It follows from the Agmon–Douglis–Nirenberg L_p-estimates that the graph norm on $\mathrm{dom}(A_p)$, $1 < p < \infty$, is equivalent to $\|\cdot\|_{2,p}$. Hence Lemma (5.2) implies that $C^2_{\mathscr{B}}(\bar{\Omega})$ *is a core for* A_p, that is, A_p is the smallest closed extension of $A_p \mid C^2_{\mathscr{B}}(\bar{\Omega})$, $1 < p < \infty$. This fact motivates the definition of the L_1-realization of $(\mathscr{A}, \mathscr{B})$ given below.

Throughout this section we suppose that the formally adjoint BVP $(\mathscr{A}^{\#}, \mathscr{B}^{\#})$ *is a regular elliptic BVP.*

(9.1) LEMMA. $\mathscr{A} \mid C^2_{\mathscr{B}}(\bar{\Omega})$ *is closable in* $L_1(\Omega)$.

PROOF. Let (u_j) be a sequence in $C^2(\bar{\Omega})$ such that $u_j \to 0$ and $\mathscr{A}u_j \to v$ in $L_1(\Omega)$. Then, by Green's formula,

$$\langle w, v \rangle = \lim_{j\to\infty} \langle w, \mathscr{A}u_j \rangle = \lim_{j\to\infty} \langle \mathscr{A}^{\#} w, u_j \rangle = 0$$

for every $w \in \mathscr{D}(\Omega)$. Hence $v = 0$, which implies the assertion. □

We define now the L_1-*realization* A_1 of $(\mathscr{A}, \mathscr{B})$ to be the closure of $\mathscr{A} \mid C^2_{\mathscr{B}}(\bar{\Omega})$ in $L_1(\Omega)$. Thus A_1 *is a closed densely defined linear operator in* $L_1(\Omega)$, *and*

$$A_1 \mid L_p(\Omega) \cap \mathrm{dom}(A_1) = A_p \tag{9.1}$$

for all $p > 1$.

The following proposition generalizes a corresponding regularity result of Brézis and Strauss [6, theorem 8 and lemma 23]. Here $D(A_1)$ denotes the Banach space $(\mathrm{dom}(A_1), \|\cdot\|_{A_1})$, where $\|\cdot\|_{A_1}$ is the graph norm.

(9.2) PROPOSITION. $D(A_1) \hookrightarrow W^1_{q,0}(\Omega)$ *for* $1 \leqq q < n/(n-1)$.

PROOF. For $1 < q < \infty$ and $u \in C(\bar{\Omega})$ let

$$\|u\|_{-1,q} := \sup\{|(u \mid v)_{L_2}| / \|v\|_{1,q} \mid v \in C^2_{\mathscr{B}^{\#}}(\bar{\Omega})\},$$

and fix any $\lambda \in \rho(-A)$. Then, by the arguments of the proof of [2, proposition 3.3], we deduce the existence of a constant c_q such that

$$\|u\|_{1,q} \leqq c_q \|(\lambda + A_1)u\|_{-1,q} \qquad \forall u \in C^2_{\mathscr{B}}(\bar{\Omega}). \tag{9.2}$$

Since $W^1_{q'}(\Omega) \hookrightarrow L_\infty(\Omega)$ for $q' > n$, that is, for $q < n/(n-1)$, it follows that

$$\|v\|_{-1,q} \leqq c'_q \sup\{|(v \mid w)_{L_2}| / \|w\|_\infty \mid w \in C^2_{\mathscr{B}^{\#}}(\bar{\Omega})\} \leqq c'_q \|v\|_1 \tag{9.3}$$

for some constant c'_q. Now the assertion follows from (9.2). □

(9.3) COROLLARY. *A_1 has a compact resolvent and $\sigma(A_1) = \sigma(A)$.*

PROOF. Fix $\lambda \in \rho(A)$ and $q \in (1, n/(n-1))$. Then (9.2) and (9.3) imply the existence of a constant c such that

$$\| u \|_1 \leqq \| u \|_{1,q} \leqq c \| (\lambda + A_1) u \|_1 \qquad \forall u \in \operatorname{dom}(A_1). \tag{9.4}$$

Hence $\lambda + A_1$ is injective. Let $v \in L_1(\Omega)$ be given, choose a sequence $\varphi_j \in \mathscr{D}(\Omega)$ such that $\varphi_j \to v$ in $L_1(\Omega)$, and let $u_j := (\lambda + A_1)^{-1} \varphi_j = (\lambda + A_p)^{-1} \varphi_j$, where $p \in (1, \infty)$ is arbitrary. Then, by (9.4),

$$\| u_j - u_k \|_1 \leqq c \| \varphi_j - \varphi_k \|_1 \qquad \forall j, k \in \mathbf{N}.$$

Thus there is a $u \in L_1(\Omega)$ such that $u_j \to u$ and $(\lambda + A_1) u_j \to v$ in $L_1(\Omega)$. This shows that $u \in \operatorname{dom}(A_1)$ and $(\lambda + A_1) u = v$, that is, $\lambda + A_1$ is surjective. Consequently, $\rho(A_1) \supset \rho(A)$, and A_1 has a compact resolvent since

$$D(A_1) \hookrightarrow W_q^1(\Omega) \overset{c}{\hookrightarrow} L_1(\Omega).$$

Since, trivially, $\sigma(A) \subset \sigma(A_1)$ by (9.1), it follows that $\sigma(A) = \sigma(A_1)$. □

It is now easy to prove the following basic

(9.4) THEOREM. $A_1' = A_\infty^{\#}$.

PROOF. Let $v \in \operatorname{dom}(A_\infty^{\#})$, $u \in C^2_{\mathscr{B}}(\bar{\Omega})$, and $1 < p < \infty$. Then

$$\langle v, A_1 u \rangle = \langle v, A_p u \rangle = \langle A_p' v, u \rangle = \langle A_{p'}^{\#} v, u \rangle = \langle A_\infty^{\#} v, u \rangle$$

by Theorem (7.1) and the definition of $A_\infty^{\#}$. Hence $A_1' \supset A_\infty^{\#}$ by Lemma (5.2). If $\lambda \in \rho(-A)$, then $\lambda + A_1' = (\lambda + A_1)'$ and $\lambda + A_\infty^{\#}$ are both isomorphisms. Thus A_1' cannot be a proper extension of $A_\infty^{\#}$. □

10. Semigroups in $L_1(\Omega)$

In the following we let $\tilde{\mathscr{A}} := -\Delta$ and $\tilde{\mathscr{B}} := (\gamma_0, \nu^j \gamma_1 \circ D_j)$, that is, $\tilde{\mathscr{B}} u = u$ on Γ_0 and $\tilde{\mathscr{B}} u = \partial u / \partial \nu$ on Γ_1. For completeness we give a simple proof of the following lemma, although it is a special case of the results of Brézis and Strauss [6].

(10.1) LEMMA. $-\tilde{A}_1 \in \mathscr{G}(L_1(\Omega), 1, 0)$.

PROOF. Let $u \in C^2_{\tilde{\mathscr{B}}}(\bar{\Omega})$, $2 \leqq p < \infty$, and $v := \bar{u} |u|^{p-2}$. Then

$$D_j v = |u|^{p-2} (D_j \bar{u} + (p-2)(\bar{u}/|u|) D_j |u|) \quad \text{and} \quad D_j |u| = \operatorname{Re}(\bar{u} D_j u)/|u|.$$

Hence, by Gauss' theorem,

$$\operatorname{Re}\langle v, (\lambda - \Delta)u\rangle = \lambda \|u\|_p^p + \operatorname{Re}\langle D_j v, D_j u\rangle$$

$$= \lambda \|u\|_p^p + \int_\Omega |u|^{p-2}(|\nabla u|^2 + (p-2)|\nabla |u||^2)dx \geqq \lambda \|u\|_p^p.$$

From this it follows that

$$\lambda \|u\|_p \leqq \|(\lambda + \tilde{A}_p)u\|_p \tag{10.1}$$

for all $u \in \operatorname{dom}(\tilde{A}_p)$ and $\lambda > 0$. Since $(0,\infty) \subset \rho(-\tilde{A})$, the Hille–Yosida theorem implies $-\tilde{A}_p \in \mathscr{G}(L_p(\Omega), 1, 0)$. Thus, since $(\tilde{\mathscr{A}}, \tilde{\mathscr{B}})$ is formally self-adjoint, we obtain $-\tilde{A}_p \in \mathscr{G}(L_p(\Omega), 1, 0)$ for all $p \in (1,\infty)$ by Lemma (2.1) and Theorem (7.1). Thus (10.1) is true for every $p \in (1,\infty)$ and $u \in C^2_{\mathscr{B}}(\bar{\Omega})$, and, letting $p \to 1$, it follows that

$$\lambda \|u\|_1 \leqq \|(\lambda + \tilde{A}_1)u\|_1 \qquad \forall u \in C^2_{\mathscr{B}}(\bar{\Omega}).$$

Now the assertion follows from the definition of $\tilde{A}_1$, Corollary (9.3), and the Hille–Yosida theorem. □

In order to simplify the notation we denote in the following the operator $-\tilde{A}_1$ simply by Δ_1. Then we prove the following fundamental

(10.2) THEOREM. *$C_0(\bar{\Omega})$ is the Δ_1-dual of $L_1(\Omega)$, that is,*

$$C_0(\bar{\Omega}) = [L_1(\Omega)]^\bullet_{\Delta_1},$$

and $L_1(\Omega)$ is Δ_1-reflexive, that is,

$$L_1(\Omega) = [C_0(\bar{\Omega})]_{\Delta_1^\bullet}.$$

If $(\mathscr{A}^\#, \mathscr{B}^\#)$ is a regular elliptic BVP, then $A^\#$ is the Δ_1-dual of A_1 and A_1 is the $\Delta_1^\bullet$-dual of $A^\#$.

PROOF. Since, by Lemma (10.1), Δ_1 generates a contraction semigroup on $L_1(\Omega)$, it follows from Lemma (2.1) that $[L_1(\Omega)]^\bullet_{\Delta_1}$ is the closure of the domain of the dual $(\Delta_1)'$ in $[L_1(\Omega)]' = L_\infty(\Omega)$. Hence, by observing that Δ_1 is formally self-adjoint, we see from Theorem (9.4) and Lemma (8.1) that $[L_1(\Omega)]^\bullet_{\Delta_1} = C_0(\bar{\Omega})$. Now, since Δ_1 has a compact resolvent by Corollary (9.3), Theorem (2.2) implies that $L_1(\Omega)$ is Δ_1-reflexive.

If $(\mathscr{A}^\#, \mathscr{B}^\#)$ is a regular elliptic BVP, then Theorem (9.4) and Lemma (8.1) show — due to the already proven facts — that $A^\#$ (that is, the $C_0(\bar{\Omega})$-realization of $(\mathscr{A}^\#, \mathscr{B}^\#)$) is the Δ_1-dual of A_1. Since $\operatorname{dom}(A_1') \subset C_0(\bar{\Omega}) = [L_1(\Omega)]^\bullet$ by Lemma (8.1) and Theorem (9.4), the remark following Lemma (3.2) and Proposition (3.6) imply $(A_1)^{\bullet\bullet}_{\Delta_1} = A_1$, whence the last assertion. □

Observe that, by the above theorem, $\Delta_1^{\bullet} = -\tilde{A}$. In abuse of notation, we denote $-\tilde{A}$ by Δ, that is, $\Delta_1^{\bullet} = \Delta$. Consequently,

$$[L_1(\Omega)]^{\bullet}_{\Delta_1} = C_0(\bar{\Omega}) \quad \text{and} \quad [C_0(\bar{\Omega})]^{\bullet}_{\Delta} = L_1(\Omega). \tag{10.2}$$

After these preparations it is now easy to prove our main results.

In the remainder of this section we suppose that $(\mathscr{A}^{\#}, \mathscr{B}^{\#})$ is a regular elliptic BVP.

(10.3) THEOREM. *$-A_1$ generates a positive, compact, analytic semigroup on $L_1(\Omega)$, which is a contraction semigroup if $a_0^{\#} \geqq 0$ and $\beta_0^{\#} \geqq 0$.*

PROOF. By Theorem (8.2) we know that $-A^{\#}$ generates a compact, positive, analytic semigroup on $C_0(\bar{\Omega})$. Moreover, Theorem (8.3) implies that $-A^{\#}$ generates a contraction semigroup on $C_0(\bar{\Omega})$, provided $a_0^{\#} \geqq 0$ and $\beta_0^{\#} \geqq 0$. From Theorem (10.2) we know that $L_1(\Omega)$ is the $\Delta_1^{\bullet}$-dual of $C_0(\bar{\Omega})$ and $-A_1$ is the $\Delta_1^{\bullet}$-dual operator of $-A^{\#}$. It is an obvious consequence of Theorem (9.4) that $(A^{\#})' \supset A_1$. Moreover, $\sigma(A^{\#}) = \sigma(A_p^{\#}) = \sigma(A'_{p'}) = \sigma(A_{p'}) = \sigma(A_1)$ by Theorem (7.1) and the fact that the spectrum of A_p is independent of $p \in [1,\infty]$. Let now $\lambda \in \sigma(A^{\#})$ and $v \in L_1(\Omega) \subset [C_0(\bar{\Omega})]'$. Then there is a unique $u \in \mathrm{dom}(A_1)$ such that $v = (\lambda + A_1)u$. Hence $v = (\lambda + (A^{\#})')u$, since $(A^{\#})' \supset A_1$, which shows that $R(\lambda, (-A^{\#})')(L_1(\Omega)) \subset L_1(\Omega)$. Since $L_1(\Omega) = C_0(\bar{\Omega})^{\bullet}_{\Delta}$ by (10.2), the assertion follows now by applying Theorem (3.5) and Theorem (3.3) with $X := C_0(\bar{\Omega})$ and $B := -A^{\#}$. □

Our next theorem shows that the semigroup e^{-tA_1} is — for each fixed $t \geqq 0$ — the continuous extension of e^{-tA_p} for every $p \in (1,\infty)$. In addition it gives a useful and natural characterization of the dual semigroup of e^{-tA_1}.

(10.4) THEOREM. *$e^{-tA_p} = e^{-tA_1} \mid L_p(\Omega)$ for $1 < p < \infty$, and $(e^{-tA_1})' = e^{-tA_1^{\#}} \mid L_\infty(\Omega)$ for every $t \geqq 0$.*

PROOF. The first assertion follows from (9.1). Since $\{e^{-tA_p} \mid t \geqq 0\}$ is an analytic semigroup on $L_p(\Omega)$, it follows that $e^{-tA_1}(L_p(\Omega)) \subset \mathrm{dom}(A_p)$ for $t > 0$ and every $p \in [1,\infty)$. Thus, by Sobolev's imbedding theorem, $e^{-tA_1^{\#}} \mid L_\infty(\Omega) \in \mathscr{L}(L_\infty(\Omega))$.

For $u, v \in L_\infty(\Omega)$ we deduce now from Theorem (7.1) and Lemma (2.1) that

$$\langle e^{-tA_1^{\#}} u, v \rangle = \langle e^{-tA_{p'}^{\#}} u, v \rangle = \langle e^{-tA'_p} u, v \rangle$$

$$= \langle (e^{-tA_p})' u, v \rangle = \langle u, e^{-tA_p} v \rangle = \langle u, e^{-tA_1} v \rangle,$$

where $1 < p < \infty$. Now the second assertion follows from the density of $L_\infty(\Omega)$ in $L_1(\Omega)$. □

Our last result of this section is concerned with the smoothing property of the semigroup $\{e^{-tA_1} \mid t \geqq 0\}$. For this we denote by $W^2(\Omega)$ the projective limit of the spaces $W_p^2(\Omega)$, $1 < p < \infty$,

$$W^2(\Omega) := \varprojlim W_p^2(\Omega).$$

Thus $W^2(\Omega)$ is the vector space $\bigcap\{W_p^2(\Omega) \mid 1 < p < \infty\}$ endowed with the family of seminorms $\{\|\cdot\|_{2,p} \mid 1 < p < \infty\}$. Consequently $W^2(\Omega)$ is a Fréchet space and

$$W_\infty^2(\Omega) \hookrightarrow W^2(\Omega) \hookrightarrow W_p^2(\Omega), \qquad 1 < p < \infty,$$

and $W_{\mathscr{B}}^2(\bar{\Omega})$ is a closed linear subspace of $W^2(\Omega)$ (cf. [27, §2.5] for the elementary facts about projective limits).

(10.5) COROLLARY. $(t \mapsto e^{-tA_1}) \in C((0,\infty), \mathscr{L}(L_1(\Omega), W_{\mathscr{B}}^2(\Omega)))$.

PROOF. Since $\{e^{-tA_1} \mid t \geqq 0\}$ is an analytic semigroup, it is well known that

$$(t \mapsto e^{-tA_1}) \in C((0,\infty), \mathscr{L}(L_1(\Omega), D(A_1))).$$

Now the assertion follows from Proposition (9.2), the first part of Theorem (10.4), the fact that $\{e^{-tA_p} \mid t \geqq 0\}$ is an analytic semigroup on $L_p(\Omega)$ for each $p \in (1,\infty)$, from (7.2), and by means of the semigroup property. □

Clearly, if $\mathscr{A}$, $\mathscr{B}$, and Ω have better regularity properties we get better smoothing properties of the semigroup $\{e^{-tA_1} \mid t \geqq 0\}$. For example, if Ω and all coefficients of $\mathscr{A}$ and $\mathscr{B}$ belong to class C^∞, it follows that

$$(t \mapsto e^{-tA_1}) \in C^\infty((0,\infty), \mathscr{L}(L_1(\Omega), C^\infty(\bar{\Omega}))).$$

This is a consequence of the fact that every analytic semigroup is an analytic function from $(0,\infty)$ into $\mathscr{L}(X, X_k)$, $k \in \mathbf{N}$, where X_k equals the domain of the k-th power of the infinitesimal generator endowed with the graph norm.

11. Contraction semigroups in $L_p(\Omega)$, $1 \leqq p < \infty$

Throughout this section we assume again that $(\mathscr{A}^\#, \mathscr{B}^\#)$ is a regular elliptic BVP.

First we consider the case that $-A_p$ generates a contraction semigroup in $L_p(\Omega)$ for every $p \in [1,\infty)$.

(11.1) THEOREM. *Suppose that* $a_0, a_0^\# \geqq 0$ *and* $\beta_0, \beta_0^\# \geqq 0$. *Then* $-A_p \in \mathscr{G}(L_p(\Omega), 1, 0)$ *for each* $p \in [1, \infty)$.

PROOF. It follows from Theorem (10.3) that $-A_1 \in \mathscr{G}(L_1(\Omega), 1, 0)$, and Theorems (8.3) and (10.2), together with Theorem (3.3), imply that $-A_1^\# \in \mathscr{G}(L_1(\Omega), 1, 0)$. Thus $\|e^{-tA_1}\|_1 \leqq 1$ and, by Theorem (10.4),

$$\|e^{-tA_1}\|_\infty = \|(e^{-tA_1^\#})'\|_\infty = \|e^{-tA_1^\#}\|_1 \leqq 1$$

for all $t \geqq 0$. Now the assertion is a consequence of the Riesz–Thorin theorem (e.g. [4, theorem 1.1.1]). □

(11.2) COROLLARY. *Suppose that* $\Gamma_0 = \partial\Omega$ *(Dirichlet boundary conditions). Then* $-A_p$ *generates a contraction semigroup on* $L_p(\Omega)$ *for each* $p \in [1, \infty)$ *iff* $a_0 \geqq 0$ *and* $a_0 - D_j a_j \geqq 0$.

PROOF. This follows from Theorem (11.1), formula (4.5), and Remark (8.4). □

(11.3) REMARK. Suppose that $\beta = \nu_a$. Then it follows from the considerations in Section 4 that $\beta_0^\# = \beta_0 + a_j \nu^j$. Hence, in this case, the conditions $a_0 \geqq 0$, $a_0 - D_j a_j \geqq 0$, $\beta_0 \geqq 0$, and $a_j \nu^j \geqq 0$ on Γ_1 are sufficient for $-A_p$ to generate a contraction semigroup in every $L_p(\Omega)$, $1 \leqq p < \infty$. Under slightly stronger conditions (namely, $\beta = \nu_a$, $a_0 \geqq \alpha > 0$, $a_0 - D_j a_j \geqq \alpha$, $\beta_0 = 0$, $a_j \nu^j \geqq 0$ on $\Gamma_1 := \partial\Omega$) Brézis and Strauss [6] have shown that $-A_1$ generates a strongly continuous contraction semigroup on $L_1(\Omega)$.

It is a more delicate question to characterize those BVPs $(\mathscr{A}, \mathscr{B})$ which generate contraction semigroups in each $L_p(\Omega)$, $1 \leqq p < \infty$, if $\Gamma_1 \neq \emptyset$. Even if one shows that $\beta_0 \geqq 0$ is a necessary condition for $-A$ to generate a contraction semigroup on $C_0(\bar{\Omega})$, it is not clear what this does mean for $-A_p$, since the coefficients a_j of $\mathscr{A}$ can be modified on a set of measure zero without changing the L_p-realization A_p for $1 \leqq p < \infty$. □

As a further consequence of Theorem (11.1) we obtain the following version of the weak maximum principle.

(11.3) PROPOSITION. *Suppose that* $a_0, a_0^\# \geqq 0$ *and* $\beta_0, \beta_0^\# \geqq 0$. *Then*

$$\operatorname*{ess\text{-}sup}_{\Omega} (1 + \lambda A_1)^{-1} u \leqq \|u^+\|_\infty \qquad \forall \lambda \geqq 0, \quad u \in L_1(\Omega), \tag{11.1}$$

where $u^+ := \max(u, 0)$.

PROOF. It follows from Theorem (11.1) that

$$\|(1 + \lambda A_1)^{-1} u\|_p \leqq \|u\|_p \qquad \forall \lambda \geqq 0, \quad p \in (1, \infty), \quad u \in L_\infty(\Omega).$$

Hence, letting $p \to \infty$, we find that

(11.2) $$\|(1+\lambda A_1)^{-1}u\|_\infty \leqq \|u\|_\infty \qquad \forall \lambda \geqq 0, \quad u \in L_\infty(\Omega).$$

Since $(1+\lambda A_1)^{-1} \geqq 0$ for $\lambda > 0$ by the positivity of the semigroup $\{e^{-tA_1} \mid t \geqq 0\}$ (e.g. [14, theorem 11.7.2]), we deduce from $u \leqq u^+$ that $(1+\lambda A_1)^{-1}u \leqq (1+A_1)^{-1}u^+$. Now the assertion follows from (11.2). □

Proposition (11.3) generalizes a corresponding result of Brézis and Strauss [6], where inequality (11.1) is one of the basic hypotheses under which the existence of weak solutions to semilinear equations of the form $A_1u + g(u) \ni f$ is proven in the case that $f \in L_1(\Omega)$ and g is a maximal monotone graph in $\mathbf{R}$. We leave it to the reader to apply the general results of this paper to semilinear elliptic BVPs along the lines of [6].

12. Spectral properties and growth estimates

Let X be an OBS. Then $x \in X^+$ is a *quasi-interior point* of X^+ if the linear hull of the order interval $[0,x] := \{y \in X \mid 0 \leqq y \leqq x\}$ is dense in X. If $T \in \mathscr{L}^+(X)$ then T is said to be *irreducible* if there exists a $\lambda > r(T)$, where $r(T)$ denotes the spectral radius of T, such that

(12.1) $$TR(\lambda, T)x = \sum_{k=1}^{\infty} \lambda^{-k}T^k x$$

is a quasi-interior point of X^+ for every $x > 0$ (e.g. [27, V.7.7]). If $X = L_p(\Omega)$, $1 \leqq p < \infty$, then $u \in X^+$ is a quasi-interior point of X^+ iff $u(x) > 0$ for a.e. $x \in \Omega$.

If C is a densely defined closed linear operator in an arbitrary Banach space then

$$s(C) := \sup\{\operatorname{Re}\lambda \mid \lambda \in \sigma(C)\}$$

is the *spectral bound* of C.

After these preparations we can prove some important spectral properties of A_p.

(12.1) THEOREM. *$\sigma(A)$ contains a least real eigenvalue λ_0, the* principal eigenvalue *of $(\mathscr{A}, \mathscr{B})$ and λ_0 is simple and has a positive eigenfunction $u_0 \in W^2_{\mathscr{B}}(\Omega)$ such that $u_0(x) > 0$ for $x \in \Omega \cup \Gamma_1$ and $(\partial u_0/\partial \tilde{\nu})(x) < 0$ for $x \in \Gamma_0$. Moreover, λ_0 is the only eigenvalue of A having a positive eigenfunction, and $-\lambda_0 = s(-A)$. Finally, $(\lambda + A_p)^{-1}$ is positive and irreducible for $\lambda > -\lambda_0$ and $1 < p < \infty$, and there is no eigenvalue $\lambda \neq \lambda_0$ of A with* $\operatorname{Re}(\lambda) = \lambda_0$.

PROOF. Let $s := s(-A)$. Hence $s = s(-A_p)$ for every $p \in (1, \infty)$, since $\sigma(A) = \sigma(A_p)$. Let now $\lambda > \hat{\lambda}$ and $p \in (1, \infty)$ be fixed, and let $T := (\lambda + A_p)^{-1}$. Then, by a standard bootstrap argument, for each $v \in L_p^+(\Omega) \setminus \{0\}$ there is a $k \in \mathbf{N}^*$ such that $T^k v \in W_q^2(\Omega)$, where $q > n$. Hence it follows from Theorems (6.1) and (12.1) that $TR(\mu, T)v$ is a quasi-interior point of $L_p^+(\Omega)$ for $\mu > r(T) =: r$. Hence T is irreducible and [27, app. 3.2] implies that $r > 0$, that r is a simple eigenvalue of T, that r possesses a positive eigenfunction u_0 which is a quasi-interior point of $L_p^+(\Omega)$, and that there is a positive eigenfunction $v_0 \in L_{p'}^+(\Omega)$ to the eigenvalue r of T' such that $\langle v_0, u \rangle > 0$ for all $u \in L_p^+(\Omega) \setminus \{0\}$. Suppose now that λ' is an eigenvalue of T possessing a positive eigenfunction u. Then $\lambda' \langle v_0, u \rangle = \langle v_0, Tu \rangle = \langle T' v_0, u \rangle = r \langle v_0, u \rangle$, which implies $\lambda' = r$. Thus r is the only eigenvalue of T having a positive eigenfunction.

Clearly, $A_p u_0 = ((1/r) - \lambda) u_0$, which shows that $\sigma(A) \cap \mathbf{R} \neq \emptyset$. Hence A has a least real eigenvalue, λ_0, and a bootstrap argument shows that $u_0 \in W_{\mathscr{B}}^2(\Omega)$. Thus, by adding the term μu_0 to both sides of the last equation, where μ is sufficiently large, we deduce from Theorem (6.1) that $u_0(x) > 0$ for $x \in \Omega \cup \Gamma_1$ and $(\partial u_0 / \partial \breve{\nu})(x) < 0$ for $x \in \Gamma_0$.

We can now invoke [13, theorem 3.3] to obtain $s \in \sigma(-A)$ and $(\lambda + A_p)^{-1} \geqq 0$ for $\lambda > s$. Hence $\lambda_0 = -s$. Let $\lambda > -\lambda_0$ be fixed and let $\mu := \lambda + \lambda_0 > 0$. Then $\sigma(\lambda + A) \subset \{z \in \mathbf{C} \mid \operatorname{Re} z \geqq \mu\}$ and $T := (\lambda + A_p)^{-1}$ is compact. Moreover, $r := r(T) \in \sigma(T)$ by the above considerations, which implies $r = 1/\mu$. From this we deduce that the positive eigenfunction u_0 of A belongs to the eigenvalue λ_0. Now all the remaining assertions, but the very last one, follow from the spectral mapping theorem (e.g. [9, lemma VII.9.2]).

Suppose that there is a $\lambda \in \sigma(A) \setminus \{\lambda_0\}$ with $\operatorname{Re}(\lambda) = \lambda_0$. Then it follows from [12, theorem (2.4)] that $\lambda_0 + ik \operatorname{Im} \lambda \in \sigma(A)$ for every $k \in \mathbf{Z}$. Since $-A_p$ generates an analytic semigroup, it is known that there are constants $\gamma \in \mathbf{R}$ and $\alpha \in (0, \pi/2)$ such that $\sigma(A) \subset \{z \in \mathbf{C} \mid |\arg z| < \alpha\}$, which gives a contradiction.

□

Since there is no assumption on the sign of β_0, the above theorem is new, even in the case of smooth coefficients of $(\mathscr{A}, \mathscr{B})$ and a smooth domain Ω.

Throughout the remainder of this section we presuppose again that $(\mathscr{A}^{\#}, \mathscr{B}^{\#})$ is a regular elliptic BVP.

By Corollary (9.3) we know that $\sigma(A_1) = \sigma(A)$. The following proposition complements Theorem (12.1).

(12.2) PROPOSITION. *$(\lambda + A_1)^{-1}$ is positive and irreducible for $\lambda > -\lambda_0$.*

PROOF. Since $(\lambda + A_1)^{-1} \supset (\lambda + A_p)^{-1}$ for $p > 1$ and $\lambda \in \rho(A)$, the positivity of $(\lambda + A_1)^{-1}$ for $\lambda > -\lambda_0$ follows from Theorem (12.1) and the density of $L_p^+(\Omega)$ in $L_1^+(\Omega)$. Now the irreducibility of $(\lambda + A_1)^{-1}$ for $\lambda > -\lambda_0$ is a consequence of Proposition (9.2) and the irreducibility of $(\lambda + A_p)^{-1}$ for $p > 1$. □

In the following theorem we give two important estimates. We emphasize the facts that these estimates are independent of $p \in [1, \infty)$ and that the exponential bound is optimal.

(12.3) THEOREM. *There are constants M and N such that*

$$\| e^{-tA_p} \|_p \leqq M e^{-\lambda_0 t} \tag{12.2}$$

and

$$\| A_p e^{-tA_p} \|_p \leqq N t^{-1} e^{-\lambda_0 t} \tag{12.3}$$

for all $t > 0$ and $p \in [1, \infty)$.

PROOF. By Theorem (12.1) and Corollary (9.3) the space $L_1(\Omega)$ has a direct sum decomposition $L_1(\Omega) = \mathbf{R}u_0 \oplus X_1$, where X_1 is invariant under A_1 and $\sigma(A_1 \mid X_1) = \sigma(A) \backslash \{\lambda_0\}$ (e.g. [15, theorem III.6.17]). Since the spectrum of A_1 is contained in a proper sector of the complex plane and there is no eigenvalue of $A_1 \mid X_1$ with real part λ_0, the fact that A_1 has a compact resolvent implies that $s_1 := s(-A_1 \mid X_1) < -\lambda_0$. The representation formula (3.3) shows that $\mathbf{R}u_0$ and X_1 are both invariant under e^{-tA_1}, that $e^{-tA_1} \mid X_1 = e^{-tA_1|X_1}$, and that $e^{-tA_1}u_0 = e^{-\lambda_0 t}u_0$ for all $t \geqq 0$. Clearly $\{e^{-tA_1|X_1} \mid t \geqq 0\}$ is a compact analytic semigroup on X_1. Hence, by the spectral mapping theorem (for the point spectrum) of analytic semigroups (e.g. [23, corollary II.3.4]), $\sigma(e^{-tA_1|X_1}) = e^{-t\sigma(A_1|X_1)}$ for every $t > 0$. Thus

$$r(e^{-tA_1|X_1}) = \sup\{|\lambda| \mid \lambda \in \sigma(e^{-tA_1|X_1})\} = e^{ts_1}$$

for $t \geqq 0$. On the other hand it is well known that $r(e^{-tA_1|X_1}) = e^{\omega_0 t}$ for all $t \geqq 0$, where

$$\begin{aligned} \omega_0 &= \lim_{t\to\infty} t^{-1} \log \| e^{-tA_1|X_1} \|_1 \\ &= \inf\{\omega \in \mathbf{R} \mid \exists M \geqq 1 : \| e^{-tA_1|X_1} \|_1 \leqq M e^{\omega t} \ \forall t \geqq 0\} \end{aligned}$$

(e.g. [8, theorem 1.22] and [7, proposition 1.1.2]). Hence there is a constant $M_1 \geqq 1$ such that $\| e^{-tA_1|X_1} \|_1 \leqq M_1 e^{-\lambda_0 t}$ for $t > 0$. This implies the existence of a constant $M \geqq 1$ such that

$$\|e^{-tA_1}\|_1 \leqq Me^{-\lambda_0 t} \qquad \forall t \geqq 0,$$

that is, such that $-A_1 \in \mathcal{G}(L_1(\Omega), M, -\lambda_0)$.

By applying the same argument to $-A$ (and by increasing M if necessary), we find that $-A \in \mathcal{G}(C_0(\bar{\Omega}), M, -\lambda_0)$. Thus $-A^{\#} \in \mathcal{G}(L_1(\Omega), M, -\lambda_0)$ by Theorem (10.2) and Theorem (3.3). Now (12.2) follows from the Riesz–Thorin theorem and from Theorem (10.4), since

$$\|e^{-tA_1}\|_\infty = \|(e^{-tA_1^{\#}})'\|_\infty = \|e^{-tA_1^{\#}}\|_1 \leqq Me^{-\lambda_0 t}$$

for $t \geqq 0$.

Since $-A_1 \in \mathcal{H}(L_1(\Omega))$ by Theorem (10.3), it follows that

$$\overline{\lim_{t\to 0}}\, t\|A_1 e^{-tA_1}\|_1 < \infty$$

(e.g. [7, proposition 1.1.11]). Hence there are constants N_1 and $\delta > 0$ such that $t\|A_1 e^{-tA_1}\|_1 \leqq N_1$ for $0 < t \leqq \delta$. Thus, by (12.2),

$$\|A_1 e^{-tA_1}\|_1 = \|e^{-(t-\delta)A_1} A_1 e^{-\delta A_1}\|_1 \leqq \delta^{-1} N_1 M e^{-(t-\delta)\lambda_0}$$

for $t \geqq \delta$. Consequently there is a constant N such that

$$\|A_1 e^{-tA_1}\|_1 \leqq Nt^{-1} e^{-\lambda_0 t} \qquad \forall t > 0.$$

By applying the same argument to $-A^{\#} \in \mathcal{H}(L_1(\Omega))$ we can assume that also

$$\|A_1^{\#} e^{-tA_1^{\#}}\| \leqq Nt^{-1} e^{-\lambda_0 t} \qquad \forall t > 0.$$

Using the fact that $A_1^{\#} \mid L_{p'}(\Omega) = A_{p'}^{\#} = A_p'$ for $1 < p < \infty$, that A_p commutes with e^{-tA_p}, and the density of $\operatorname{dom}(A_p)$ in $L_1(\Omega)$, we see, similarly as in the proof of Theorem (10.4), that

$$A_1 e^{-tA_1} \mid L_\infty(\Omega) = (A_1^{\#} e^{-tA_1^{\#}})'$$

for $t > 0$. Now (12.3) follows again by the Riesz–Thorin theorem. □

The following corollary shows that the semigroup $\{e^{-tA_p} \mid t \geqq 0\}$ has a holomorphic extension to a sector around $\mathbf{R}^+$ which is independent of $p \in [1, \infty)$.

(12.4) COROLLARY. *There exists $\alpha \in (0, \pi/2]$ such that $\{e^{t\lambda_0} e^{-tA_p} \mid t \geqq 0\} \subset \mathcal{L}(L_p(\Omega))$ extends for each $p \in [1, \infty)$ to a bounded holomorphic semigroup on the sector $\{z \in \mathbf{C} \mid |\arg z| < \alpha\}$.*

PROOF. It follows from (12.2) and (12.3) that there is a constant M_1 such that

$$\|(A_p - \lambda_0) e^{-t(A_p - \lambda_0)}\|_p \leqq e^{-\lambda_0 t}\{\|A_p e^{-tA_p}\|_p + |\lambda_0|\, \|e^{-tA_p}\|_p\} \leqq M_1/t$$

for $0 < t \leqq 1$ and $1 \leqq p < \infty$. Now the assertion is a consequence of well known facts about analytic semigroups (e.g. [7, proposition 1.1.11]). □

Since $W^2_{\mathscr{B}}(\Omega) \hookrightarrow L_\infty(\Omega)$, it follows from Theorem (10.4) and Corollary (10.5) that $e^{-tA_1} \in \mathscr{L}(L_p(\Omega), L_q(\Omega))$ for $1 \leqq p \leqq q \leqq \infty$ and $t > 0$. Our last proposition gives an estimate for the $\|\cdot\|_{p,q}$-norm, that is, the norm in $\mathscr{L}(L_p(\Omega), L_q(\Omega))$ of e^{-tA_1}.

(12.5) PROPOSITION. *Let $1 \leqq p < q \leqq \infty$. Then there is a constant $c := c(p,q)$ such that*

$$\|e^{-tA_1}\|_{p,q} \leqq ct^{-(n/2)(1/p-1/q)}e^{-\lambda_0 t} \qquad \forall t > 0.$$

PROOF. By replacing A_1 by $A_1 - \lambda_0$, we can assume that $\lambda_0 = 0$. Let $\alpha := \frac{1}{2}n(1/p - 1/q)$ and assume that $p > 1$. If $\alpha < 1$, then

$$\|u\|_q \leqq c\|u\|^{\alpha}_{2,p}\|u\|^{1-\alpha}_p$$

by the Gagliardo–Nirenberg inequality (e.g. [11, theorem 10.1]). Since $\|\cdot\|_{2,p}$ is equivalent to the graph norm of $A_p = A_1 \,|\, L_p(\Omega)$ and since A_p is invertible, the assertion follows from Theorem (12.3). If $\alpha \geqq 1$, we obtain the assertion by an obvious iteration argument based on the semigroup property. Thus, in particular,

$$\|e^{-tA_1}\|_{p,\infty} \leqq ct^{-n/2p} \qquad \forall t > 0.$$

By applying this argument to $e^{-tA^{\#}_q}$ and using Theorem (10.4) we see that

$$\|e^{-tA_1}\|_{1,q} = \|e^{-tA^{\#}_1}\|_{q',\infty} \leqq ct^{-(n/2)(1-1/q)t} \qquad \forall t > 0$$

and the assertion has been proven. □

Estimates of the above type play a considerable rôle in connection with semilinear evolution equations (e.g. [26, 36]). The proof of Proposition (12.5) for $p > 1$ follows [35, lemma 4.1].

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